Chapter 5 : Vector spaces

Def 1. (Vector space)

A vector space V is a set of vectors with two operations defined, addition and scalar multiplication which satisfy the axioms of addition and scalar multiplication

Def 2 (Axioms of Addition)

Let $\vec{v}, \vec{w}, \vec{z}$ be vector space V. Then they satisfy the following the following of addition:

- Closed under addition: If $\vec{v}, \vec{w}$ are in V, then $\vec{v}+\vec{w}$ is also in V.
- The Commutative Law of Addition.
$$\vec{v}+\vec{w}=\vec{w}+\vec{v}$$
- The Associative Law of Addition.
$$(\vec{v}+\vec{w})+\vec{z}=\vec{v}+(\vec{w}+\vec{z})$$
- The Existence of an Additive Identity.
$$\vec{v}+\vec{0}=\vec{v}$$
- The Existence of an Additive Inverse
$$\vec{v}+(-\vec{v})=\vec{0}$$

Def 3 (Axioms of Scalar Multiplication)

Let $a, b \in \mathbb{R}$ and let $\vec{v}, \vec{w}, \vec{z}$ be vectors in a vector space V. Then they satisfy the following axioms of scalar multiplication:

- Closed under ~~addition~~ scalar multiplication

If a is a real number, and $\vec{v}$ is in V, then $a\vec{v}$ in V.

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- $a(\vec{v}+\vec{w}) = a\vec{v} + a\vec{w}$
- $(a+b)\vec{v} = a\vec{v} + b\vec{v}$
- $a(b\vec{v}) = (ab)\vec{v}$
- $1\vec{v} = \vec{v}$.

Ex 1 $\mathbb{R}^n$ is a vector space.

Solution $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$

- $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ $\vec{x}+\vec{y} = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix} = \begin{bmatrix} y_1+x_1 \\ \vdots \\ y_n+x_n \end{bmatrix} = \vec{y}+\vec{x}$

- $(\vec{x}+\vec{y})+\vec{z} = \vec{x}+(\vec{y}+\vec{z})$

- Next, we show the existence of an additive identity.

Let $\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ $\vec{x}+\vec{0} = \vec{x}$

Hence the zero vector $\vec{0}$ is an additive identity.

- Next, we prove the existence of an additive inverse.

Let $-\vec{x} = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$ $\vec{x}+(-\vec{x}) = \vec{0}$

Hence $-x$ is a additive inverse

$a\vec{x} = a\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} ax_1 \\ \vdots \\ ax_n \end{bmatrix}$ we first show that $\mathbb{R}^n$ is closed under scalar multiplication.

- $a(\vec{x}+\vec{y}) = a\vec{x} + a\vec{y}$
- $(a+b)\vec{x} = a\vec{x}+b\vec{x}$
- $a(b\vec{x}) = (ab)\vec{x}$
- $1\vec{x} = \vec{x}$

Ex2 (Vector space of polynomials)

Let $\mathbb{P}_2$ be the set of all polynomials of a most degree 2 as well as the zero polynomial. ~~...~~ then $\mathbb{P}_2$ is a vector space.

solution We can write $\mathbb{P}_2 = \{a_2x^2 + a_1x + a_0 \mid a_i \in \mathbb{R} \text{ for all } i\}$

To show that $\mathbb{P}_2$ is a vector space, we verify the axioms. Let $p(x)$, $q(x)$ and $r(x)$ be polynomials in $\mathbb{P}_2$ and let a, b, c be real numbers. write $p(x) = p_2x^2 + p_1x + p_0$ $q(x) = q_2x^2 + q_1x + q_0$ and $r(x) = r_2x^2 + r_1x + r_0$.

- $p(x) + q(x) = (p_2 + q_2)x^2 + (p_1 + q_1)x + (p_0 + q_0) \in \mathbb{P}_2$

  $\mathbb{P}_2$ is closed under addition
- $(p(x) + q(x)) + r(x) = p(x) + (q(x) + r(x))$
- Let $0(x) = 0x^2 + 0x + 0$ ($\exists$ an additive identity)

  $p(x) + 0(x) = p(x)$ Hence an additive identity exist.
- we must prove that exists an additive inverse. Let $-p(x) = -p_2x^2 - p_1x - p_0$. and consider the following:

  $p(x) + (-p(x)) = 0(x)$. Hence an additive inverse $-p(x)$ exists.

→ We need to verify the axioms related to scalar multiple

- $a(p(x) + q(x)) = a\,p(x) + a\,q(x)$.
- $(a+b)\,p(x) = a\,p(x) + b\,q(x)$.
- $a(b\,p(x)) = (ab)\,p(x)$
- $1.p(x) = p(x)$ where $1(x) = 1$.

Hence $\mathbb{P}_2$ is the vector space.

Ex3 Let $M_{2,3}$ be the set of all (2×3) matrices.
show that $M_{2,3}$ is a vector space.

Ex4 (vector space of Function)
Let S be a nonempty set and define F_s to be the set of real function on S. $F_s : S \longrightarrow \mathbb{R}$
Let a, b, c be scalar and f, g, h functions, the vector operation are defined as

$$(f+g)(x) = f(x) + g(x)$$
$$(af)(x) = a(f(x))$$

Show that F_s is a vector space.

Solution To verify F_s is a vector space, we must prove the axioms beginning with those for addition. Let f, g, h be functions in F_s.

- $(f+g)(x) = f(x) + g(x)$ $\quad F_s$ is under addition
- $(f+g)(x) = f(x) + g(x) = (g+f)(x)$ $\quad$ commutative law of addition $\forall x \in S$.
- $((f+g)+h)(x) = (f+g)(x) + h(x) = f(x) + g(x) + h(x) = (f+(g+h))$
  $\Rightarrow (f+g)+h = f+(g+h)$ (associative law)
- Let 0 denote the function which is given $0(x) = 0$

Then this is additive identity because

$$(f+0)(x) = f(x) + 0(x) = f(x)$$

and so $f + 0 = f$

- Check for an additive inverse, Let $-f$ be the functions which satisfies $(-f)(x) = -f(x)$.

Then $\big(f+(-f)\big)(x) = f(x) + (-f(x)) = f(x) - f(x) = 0$

Hence $f + (-f) = 0$

- Check the axioms for multiplication.

- $\big((a+b)f\big)(x) = (a+b)f(x) = af(x) + bf(x) = (af+bf)(x)$
  and so $(a+b)f = af + bf$.

- $\big(a(f+g)\big)(x) = a(f+g)(x) = a\big(f(x)+g(x)\big) = af(x) + ag(x)$
  $= (af+ag)(x)$ and so $a(f+g) = af + ag$

- $\big((ab)f\big)(x) = (ab)f(x) = a\big(bf(x)\big) = \big(a(bf)\big)(x)$
  so $(abf) = a(bf)$

- $(1f)(x) = 1f(x) = f(x)$ so $1f = f$.

<u>Theorem</u> In any vector space, the following are true.

1. $\vec{0}$, the additive identity, is unique
2. $-\vec{x}$, the additive inverse, is unique
3. $0\vec{x} = \vec{0}$ for all vector $\vec{x}$
4. $(-1)\vec{x} = -x$ for all vector $\vec{x}$

<u>proof</u> ~~...~~ 1) suppose $\vec{0}'$ is also additive identity. Then

$\vec{0} + \vec{0}' = \vec{0}$

By commutative property $0 = 0 + 0' = 0' + 0 = 0'$

So $0 = 0'$ then $\vec{0} = \vec{0}'$ $\vec{0}$ is unique

2) $\vec{x} + \vec{y} = 0 \Rightarrow \vec{y} = 0 + \vec{y}' = (-\vec{x} + \vec{x}) + \vec{y} = -\vec{x} + (\vec{x} + \vec{y})$

3. $0\vec{x} = (0+0)\vec{x} = 0\vec{x} + 0\vec{x}$ $\qquad 0 = 0+0$

and $\underbrace{0\vec{x} + (-0\vec{x})}_{0} = \underbrace{0\vec{x} + \underbrace{0\vec{x} + (-0\vec{x})}}$

$$0 = 0x + 0$$

$$\Rightarrow 0\vec{x} = \vec{0}.$$

4. $(-1)\vec{x} + \vec{x} = (-1)\vec{x} + 1\vec{x} = (-1+1)\vec{x} = 0\vec{x} = \vec{0}$

$$\Rightarrow (-1)\vec{x} = -\vec{x}.$$

<u>Theorem</u> Let V be a vector space. Then $\vec{v} + \vec{w} = \vec{v} + \vec{z}$ implies that $\vec{w} = \vec{z}$ for all $\vec{v}, \vec{w}, \vec{z} \in V$.

2- Spanning Sets

Def 1 (Subset)

Let X and Y be two sets. If all elements of X are also elements of Y then we say that X is subset of Y.

we write $X \subseteq Y$.

Def 3 (Linear combination)

Let V be a vector space and let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \leq V$.

A vector $\vec{v} \in V$ is called a linear combination of the v_i if there exists scalars $c_i \in \mathbb{R}$ such that

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$$

Def (Span of vector)

Let $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$ then

$$\text{span}\{\vec{v}_1, \ldots, \vec{v}_n\} = \left\{ \sum_{i=1}^{n} c_i \vec{v}_i \; ; \; c_i \in \mathbb{R} \right\}$$

(Collection of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a spanning set for V if $V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$).

Ex1 Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Determine if A and B are in $\text{span}\{M_1, M_2\} = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

<u>Solution</u>. we want to see if s, t scalars such that $A = sM_1 + tM_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The solution to this equation is given by

$$1 = s$$
$$2 = t$$

A is in $\text{span}\{M_1, M_2\}$

Consider B. $B = sM_1 + tM_2$. s, t scalar

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

no values of s and t can found.

B is not $\text{span}\{M_1, M_2\}$.

Ex2 (polynomial span)

Show that $p(x) = 7x^2 + 4x - 3$ is span $\{4x^2 + x, x^2 - 2x + 3\}$

<u>Solution</u> $7x^2 + 4x - 3 = a(4x^2 + x) + b(x^2 - 2x + 3)$

$$\rightarrow \begin{cases} 4a+b=7 \\ a-2b=4 \\ 3b=-3 \end{cases} \Rightarrow \begin{matrix} a=2 \\ b=-1 \end{matrix}$$

$$\Rightarrow 7x^2+4x-3 = 2(4x^2+x)-(x^2-2x+3)$$

Hence $p(x)$ is the given span.

Ex3 $S=\{x^2+1,\ x-2,\ 2x^2-x\}$ show that S is a span set for P_2, the set of all polynomials of degree at mo

Solution $p(x)=ax^2+bx+c$

$$p(x)= ax^2+bx+c = r(x^2+1)+s(x-2)+t(2x^2-x)$$

$$ax^2+bx+c = (r+2t)x^2-(s-t)x+(r-2s)$$

$$a = r+2t$$
$$b = s-t \qquad (r,s,t)$$
$$c = r-2s$$

to check that a solution exists, set up the augmented matrix and row reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & a \\ 0 & 1 & -1 & b \\ 1 & -2 & 0 & c \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}a+2b+\frac{1}{2}c \\ 0 & 1 & 0 & \frac{1}{4}a-\frac{1}{4}c \\ 0 & 0 & 1 & \frac{1}{4}a-b-\frac{1}{4}c \end{array}\right]$$

Hence S is a spanning set for P_2

3. Linear Independence

Def 1 (Linear independence)
Let V be a vector space. If $\{\vec{v}_1, \ldots, \vec{v}_n\} \subseteq V$
Then it is linearly independent if

$$\sum_{i=1}^{n} a_i \vec{v}_i = \vec{0} \text{ implies } a_1 = \cdots = a_n = 0$$

where the a_i are numbers

Def (Linear dependent)
Let $S = \{\vec{v}_1, \ldots, \vec{v}_n\} \subset V$, a vector space, is linearly dependent if there are scalars $a_1, \ldots, a_n$ not all zero for which $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$

Ex 1 (Linear independence)
Let $S \subseteq P_2$ be a set of polynomial given by

$$S = \{x^2 + 2x - 1,\ 2x^2 - x + 3\}$$

Determine if S is linearly independent.

solution To determine if this S is linearly independent, we write $a(x^2 + 2x - 1) + b(2x^2 - x + 3) = 0$
If it is linearly independent then $a = b = 0$??

$$a(x^2 + 2x - 1) + b(2x^2 - x + 3) = 0x^2 + 0x + 0$$

$$ax^2 + 2ax - a + 2bx^2 - bx + 3b = 0x^2 + 0x + 0$$

$$(a + 2b)x^2 + (2a - b)x - a + 3b = 0x^2 + 0x + 0$$

$a+2b=0$ The augmented matrix and resulting
$2a-b=0$ reduced row-echelon form are given by
$-a+3b=0$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & -1 & 0 \\ -1 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Hence the solution $a=b=0$ and the set is linearly independent

Ex2 (Dependent set)
Determine if the set S given below is independent.

$$S=\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\3\\5\end{bmatrix}\right\}$$

Solution $a\begin{bmatrix}-1\\0\\1\end{bmatrix}+b\begin{bmatrix}1\\1\\1\end{bmatrix}+c\begin{bmatrix}1\\3\\5\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\Rightarrow a=2$ $b=3$ and $c=-1$ therefore S is dependent

~~Ex Let V be vector space and suppose $S\subset V$ is a set of linearly independent vectors given by $S=\{\vec{u},\vec{v},\vec{w}\}$~~

Ex $R=\{2\vec{u}-\vec{w}, \vec{w}+\vec{v}, 3\vec{v}+\frac{1}{2}\vec{u}\}$ is linearly independent

sol $a(2\vec{u}-\vec{w})+b(\vec{w}+\vec{v})+c(3\vec{v}+\frac{1}{2}\vec{u})=\vec{0}$

$2a\vec{u}-a\vec{w}+b\vec{w}+b\vec{v}+3c\vec{v}+\frac{1}{2}c\vec{u}=\vec{0}$

$(2a+\frac{1}{2}c)\vec{u}+(b+3c)\vec{v}+(-a+b)\vec{w}=\vec{0}$

$$\left[\begin{array}{ccc|c} 2 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 3 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right] \dashrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Rightarrow a=b=c=0.$$

Th If $S=\{v_1, v_2, \dots, v_n\}$ is linearly independent then every subset of S is ~~the~~ linearly independent.

4. Sub spaces

Let V be a vector space and $U \subset V$. We will call U a subspace of V if U is closed under vector addition, scalar multiplication and satisfies all of the vector space axioms.

- V is nonempty
- the linear combination $a\vec{u} + b\vec{v}$ is also in V. / or $\vec{u} + \vec{v} \in V$, $a\vec{u} \in V$

Ex $V = \mathbb{R}^3 = \{ (a,b,c) \mid a,b,c \in \mathbb{R} \}$

$$U = \{ (a,b,0) \mid a,b \in \mathbb{R} \}$$

$U \subset V$ and U is a sub space of.

Th Let V be a vector space and $S = \{ v_1, v_2, \dots, v_k \} \subset V$. Then $\text{span}(v_1, \dots, v_n)$ is a ~~linear~~ subspace of V

proof $S = \text{Span}(v_1, \dots, v_n) = \left\{ \sum_{i=1}^{k} a_i v_i \mid a_i \dots a_k \in \mathbb{R} \right\}$

$u \in S \Rightarrow u = a_1 v_1 + \dots + a_n v_n$

$w \in S \Rightarrow w = b_1 v_1 + \dots + b_n v_n$

$$\left. \begin{aligned} v + w &= (a_1 + b_1) v_1 + \dots + (a_k + b_k) v_k \\ cv &= ca_1 v_1 + \dots + ca_n v_n \end{aligned} \right\} \in V$$

Ex (Product spaces)

Let V and W be vector spaces defined over the same field. We define the new vector space $Z = V \times W$

$$Z = \{ (v, w) \mid v \in V, w \in W \}.$$

5. <u>Bases</u> $\quad G(v_1, \dots v_k) = \left\{ \sum_1^k a_i v_i \mid a_i \in \mathbb{R} \right\}$

<u>Def</u> Let V be a Vector space and $S = \{v_1, \dots, v_n\} \subset$
We call S a basic of V if S is linearly independent for V

<u>Ex</u> $V = \mathbb{R}^3$ $\quad S = \{e_1, e_2, e_3\}$ then S is a basic for V.

<u>Solution</u> V is spanned by S. Now suppose that

$$0 = a_1 e_1 + a_2 e_2 + a_3 e_3$$

<u>or</u> $\quad (0,0,0) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$$= (a_1, a_2, a_3) \Rightarrow a_1 = a_2 = a_3 = 0$$

Thus the set $\{e_1, e_2, e_3\}$ is linearly independent.

<u>Ex</u> $\quad S = \{v_1, v_2\} = \{(1,0,1), (1,-1,0)\} \subset \mathbb{R}^3$
S is linearly independent and therefore a basic of $G(S)$

System of Generators

1

Def Let $x_1, \dots, x_n \in E$. An element x of E is said to be linear combination of $x_1, \dots, x_n$ over K if $\exists\, \alpha_1, \dots, \alpha_n \in K$ such that $x = \alpha_1 x_1 + \dots + \alpha_n x_n$.

Th Let $x_1, \dots, x_n \in E$. If V denotes the set of all linear combination of $x_1, \dots, x_n$ over K, then

(i) $x_i \in V$ for all $1 \leq i \leq n$.

(ii) V is subspace of E over K

(iii) If W is a subspace of E over K containing $x_1, \dots, x_n$, then $V \subseteq W$.

Def Let W be a subspace of E and let $x_1, \dots, x_n \in W$. We say that $x_1, \dots, x_n$ form a system of generators of W (or that $x_1, \dots, x_n$ generate W) over K if W is the set of all linear combination of $x_1, \dots, x_n$ over K.

Th Let W be ~~superset of~~ subspace of E and $x_1, \dots, x_n \in W$. If $x_1, \dots, x_n$ form a system of generators of W over K and if $y_1, \dots, y_m \in W$, then $x_1, \dots, x_n, y_1, \dots, y_m$ form a system of generators of W over K.

Ex the set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a system of generators of $\mathbb{R}^3$ over $\mathbb{R}$. Then $\exists\, a, b, c \in \mathbb{R}$ such that

$x = (a, b, c)$ $\quad x = (a,0,0) + (0,b,0) + (0,0,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

hence S is a system of generators of $\mathbb{R}^3$ over $\mathbb{R}$. ©

2) the elements $x_1=(1,0,0)$, $x_2(0,1,0)$ $x_3=(1,0,0)$ $x_4=(0,0,1)$ for a system of generators of $\mathbb{R}^3$ over $\mathbb{R}$, because, we have that x_1, x_2, x_4 form a system of generators of $\mathbb{R}^3$ over $\mathbb{R}$, hence x_1, x_2, x_3, x_4 form a system of generators of $\mathbb{R}^3$ over $\mathbb{R}$.

<u>Th</u> If $u_1=(a_{11}, a_{12}, \ldots a_{1n})$ $u_2=(a_{21}, a_{22}, \ldots a_{2n})$ $\ldots$ $u_m=(a_{m1}, a_{m2}, \ldots, a_{mn})$ are element of $\mathbb{K}^n$, then they form a system of generators of K^n over K if and only if the rank of the matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & & \cdots & a_{mn} \end{bmatrix}$$ is n.

<u>Ex</u> Check if the given element form a system of generators of $\mathbb{R}^3$ over $\mathbb{R}$ or not in the following cases: (i) $x_1=(1,1,0)$ $x_2=(2,1,1)$ $x_3=(2,0,1)$

(ii) $x_1=(1,2,1)$ $x_2=(-2,3,0)$ $x_3=(-1,5,1)$

<u>solution</u> i) $A=\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ hence $\text{Rank}(A)=3$

so x_1, x_2, x_3 for a system of generators of $\mathbb{R}^3$ over $\mathbb{R}$

(ii) $A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & 3 & 0 \\ -1 & 5 & 1 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 7 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ which is a row echelon form of A

rank$(A) = 2$ and so, x_1, x_2, x_3 do not form a system of generators of $\mathbb{R}^3$ over $\mathbb{R}$.

Basic of vector space

Def Let $x_1, \dots, x_n \in E$. We say that $\{x_1, \dots, x_n\}$ is a basic of E over K if

(i) $x_1, \dots, x_n$ are linearly independent over K.
(ii) $x_1, \dots, x_n$ form a system of generators of E over K.

Ex 1) $\{(1,0), (0,1)\}$ is a basic of $\mathbb{R}^2$ over $\mathbb{R}$

$(1,0)$ $(0,1)$ is a system of generators of $\mathbb{R}^2$ because $\forall x \in \mathbb{R}^2$, $\exists a, b \in \mathbb{R}$ such that $x = (a,b) = a(1,0) + b(0,1)$

Also $(1,0), (0,1)$ are linearly independent over $\mathbb{R}$ because if $a, b \in \mathbb{R}$ then $a(1,0) + b(0,1) = (0,0) \Rightarrow (a,0) + (0,b) = (0,0)$

$\Rightarrow a = b = 0$

th if $\{x_1, \dots, x_n\}$ is a basic of E over K, then every element u of E is uniquely written in the form $u = a_1 x_1 + \dots + a_n x_n$ where $a_1, \dots, a_n \in \mathbb{K}$.

~~Sum~~ Direct Sum of Vector spaces

Def Let U, W be subspaces of V. Then V is said to be the direct sum of U and W, and we write

$V = U \oplus W$, if 1) $V = U + W$

2) $U \cap W = \{0\}$

$L(U \cap W) = 0$

Def (Sum of subspaces)

Let S and T be two subspaces of V, then their sum (linear sum) $S+T$ is the set of all sum $u+v$ such that $u \in S$ and $v \in T$. Thus $S+T = \{u+v \,;\, u \in S, v \in T\}$

Ex Consider the vector space $V = \mathbb{R}^3$. Let $S = \{(x, y, 0) \; x, y \in \mathbb{R}\}$ $T = \{(x, 0, z) \; x, z \in \mathbb{R}\}$ prove that S and T are subspaces of V such that $V = S+T$ but V is not the direct sum of S and T

solution $v = (x, y, z) = \underset{\in S}{(x, y, 0)} + \underset{\in T}{(0, 0, z)}$ $\in V$

$v \in S+T$

$\Rightarrow V \subset S+T$ and $S+T \subset V$

$\Rightarrow V = S+T$.

but $S \cap T = \{(x, 0, 0), x \in \mathbb{R}\}$ is the x-axis

$S \cap T \neq \{(0, 0, 0)\}$ V is not sum direct of S and T

Th Let $S_1, S_2, \dots, S_n$ be subspace of a vector space V. Such that $V = S_1 + S_2 + \dots + S_n$. Then, the following are equivalent:

(i) $V = S_1 \oplus S_2 \oplus \dots \oplus S_n$.

(ii) $S_1, S_2, \dots, S_n$ are independent

(iii) For each $v_i \in S_i$ $\sum_{i=1}^{n} v_i = 0 \Rightarrow v_i = 0$

Dimension of a vector space

Def The dimension of vector space V is the nb of vectors in a basic of V. We write $\dim(V)$.

Ex $\dim(\mathbb{R}^n) = n$ $\qquad \dim(\mathbb{P}_n) = n+1$

$$\mathbb{P}_n = \text{~~Span~~}\ G(1, x, \dots, x^n)$$

$$G(1, x, \dots x^n)$$